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# Basis-free representations for the stress rate of isotopic materials

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**Abstract**

In terms of the principal components of stress tensor and a scalar response function, two kinds of basis-free representation for the stress rate are obtained, which only include the time rate of the strain, tensor functions and invariants of the strain tensor. The expressions for the stress rate valid in the cases that the eigenvalues of the strain are distinct, doubly coalescent, and triply coalescent, respectively.


*Keywords:* Basis-free representation; Stress rate; Strain rate

## 1. Introduction

In continuum mechanics, tensorial physical quantities admit either principal axis representation or basis-free representation. These two types of representation are completely equivalent. The principal axis representation, expressing a tensor quantity in the frame of principal direction of a concerned tensor, follows from Hill's principal axis method (Hill, 1978, Silhavý, 1997). The basis-free representation relies on the principal invariants and powers of arguments (Rivlin and Ericksen, 1955, Truesdell and Noll, 1965).

The principal axis representation gives better control over the differentiability, invertibility and convexity, and many results in continuum mechanics can be advantageously obtained by this method, such as, spin tensor, time rate of generalized strain and conjugate stress. However, further abstract or numerical manipulations with the principal method are quite expensive. They require that not only the eigenvalues but also the eigenvectors be determined.

In order to avoid calculation of the eigenvalues and corresponding eigenvectors, once a principal axis formula is proposed, there arises naturally the problem of finding its basis-free representation. It has attracted the attention of many researchers in past decades, such as, the expressions for spin tensor (see, e.g.,

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Mehrabadi and Nemat-Nasser, 1987, Wang and Duan, 1991, Macmillan, 1992, Dui, 1999), time rate of strain (see, e.g., Wang and Duan, 1991, Man and Guo, 1993, Xiao, 1995, Xiao et al., 1998, Dui et al., 1999), and conjugate stress (see, e.g., Hoger, 1987, Guo and Man, 1992, Xiao, 1995, Dui et al., 2000). However, these methods are applicable for special cases. The basis-free representation of general isotropic functions is a more difficult problem.

For finite elastic deformation, the constitutive equations can be provided by a conjugate pair of stress and strain introduced by Hill (1968). Let $\mathbf{E}$ be the Green strain tensor, and let $\{e_i\}$ and $\{\mathbf{N}_i\}$ be eigenvalues (principal stretches) and subordinate ortho-normal eigenvectors (principal directions) of $\mathbf{E}$, respectively. If we confine our attention to isotropic materials, the constitutive equations may be described by the Green strain and its conjugate stress $\mathbf{T}$, the second Piola–Kirchhoff stress, as

$$\mathbf{T} = \sum_i t_i(e_1, e_2, e_3)\mathbf{N}_i \otimes \mathbf{N}_i \tag{1}$$

where $t_i$ are the principal values of $\mathbf{T}$.

Once the constitutive relation is proposed, based on the principal axes method, a more compact formula for the stress rate was derived with the components of the strain and its rate in the principal frame $\{\mathbf{N}_i\}$ (Hill, 1968, Ogden, 1974). Evidently the component expression of the stress rate, as given in the principal frame, is valid only in that frame.

Basis-free representation of the stress rate has an important role to play in the description of material behavior. The time rates of conjugate stress and its corresponding strain are connected by a fourth-order tensor, the elasticity tensor. Accordingly, if the basis-free expression for the stress rate is given, the basis-free expression for the elasticity tensor may be obtained immediately.

Since the stress rate is an isotropic tensor function of the strain tensor and strain rate with linear in the strain rate for the isotropic materials, it is possible in view of the representation theorem (Rivlin and Ericksen, 1955) for isotropic functions to find a basis-free expression for the time rate of the stress with the strain tensor, strain rate and their invariants.

The explicit expression of the stress rate for the Green elastic material is well studied (see, e.g. Marsden and Hughes, 1993). However, the problem of finding the basis-free expression for general isotropic materials becomes more difficult than the above ones. As a result, most early methods are not valid again for this problem. Thus far, a basis-free expression for the general stress rate has not been available.

One objective of this paper is to obtain explicit basis-free representations for the stress rate, which only include the time rate of the strain, functions of the strain tensor and their invariants. The paper is organized as follows: in Section 2, some preliminary results, which will be used in the remainder of the paper, are supplied. In Section 3, in order to avoid the difficulty in dealing three symmetric functions, a single scalar response function $f$ to express the principal stress is introduced. The expressions for the stress tensor are given by the function $f$ in three different eigenvalue cases. In Section 4, two kinds of basis-free representations for the stress tensor are given. First, based on the principal components of the stress tensor, the stress rate is expressed as the form of isotropic tensor function representation suggested by Rivlin and Ericksen (1955), and 12 coefficients are determined. Then, the basis-free representations for the stress rate are derived by the scalar function $f$ in three different eigenvalue cases. It can be seen that this expression is more concise than the first one. Finally, an application of our basis-free formulae for the stress rate is illustrated by two examples in Section 5.

## 2. Some preliminary results

Since the deformation gradient tensor $\mathbf{F}$ has the positive determinant, it admits unique positive definite second-order tensors $\mathbf{U}$ and $\mathbf{V}$, and proper orthogonal second-order tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{2}$$

where $\mathbf{U}$ and $\mathbf{V}$, respectively, are called the right and left stretch tensors. The right and left Cauchy–Green strain tensors, $\mathbf{C}$ and $\mathbf{B}$, are related to $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{F}$ by

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}} \tag{3}$$

Let $\lambda_i$ be the principal values of $\mathbf{U}$ and $\mathbf{V}$ corresponding to the principal direction $\mathbf{N}_i$ and $\mathbf{n}_i$, respectively, so that

$$\mathbf{U} = \sum_i \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{V} = \sum_i \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i, \quad \mathbf{C} = \sum_i \Lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{4}$$

and

$$\mathbf{R} = \sum_i \mathbf{n}_i \otimes \mathbf{N}_i, \quad \mathbf{F} = \sum_i \lambda_i \mathbf{n}_i \otimes \mathbf{N}_i \tag{5}$$

where $\Lambda_i = \lambda_i^2$.

Let the invariants of $\mathbf{C}$ be

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{C}) = \Lambda_1 + \Lambda_2 + \Lambda_3 \\ I_2 &= \frac{1}{2}[(\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)] = \Lambda_1\Lambda_2 + \Lambda_2\Lambda_3 + \Lambda_1\Lambda_3 \\ I_3 &= \det(\mathbf{C}) = \Lambda_1\Lambda_2\Lambda_3 \end{aligned} \tag{6}$$

The rate-type constitutive equation of elastic materials with linear or piecewise linear incremental loading in the reference configuration can be described by the Green strain

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{7}$$

and its conjugate stress, the second Piola–Kirchhoff stress tensor $\mathbf{T}$, as

$$\dot{\mathbf{T}} = \mathbf{L} : \dot{\mathbf{E}} \tag{8}$$

where the superposed dot denotes the material time rate. In Eq. (8) the fourth-order tensor is called the elasticity tensor (or first-order modulus), and may be represented symbolically as

$$\mathbf{L} = (\mathbf{T})_{,\mathbf{E}} = 2(\mathbf{T})_{,\mathbf{C}} \tag{9}$$

For isotropic elastic materials, the symmetric second Piola–Kirchhoff stress tensor $\mathbf{T}$ is expressible as a function of the Green strain tensor $\mathbf{E}$

$$\mathbf{T} = \mathbf{T}(\mathbf{E}) \tag{10}$$

and satisfies the identity

$$\mathbf{T}(\mathbf{QEQ}) = \mathbf{QT}(\mathbf{E})\mathbf{Q} \tag{11}$$

for all orthogonal $\mathbf{Q}$.

## 3. Representation of the second Piola–Kirchhoff stress tensor for isotropic material

First, assume that $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3 \neq \Lambda_1$, that is

$$\Delta_- = (\Lambda_2 - \Lambda_3)(\Lambda_3 - \Lambda_1)(\Lambda_1 - \Lambda_2) \neq 0 \tag{12}$$

If we define

$$\hat{\mathbf{C}} = (I_1^2 - 4I_2)\mathbf{I} + 2I_1\mathbf{C} - 3\mathbf{C}^2 \tag{13}$$

then it can be verified that (Dui, 1999)

$$\Delta_-^2 = \det(\hat{\mathbf{C}}) = 18I_1I_2I_3 + I_1^2I_2^2 - 4I_1^3I_3 - 4I_2^3 - 27I_3^2 \tag{14}$$

The conjugate pair $(\mathbf{T}, \mathbf{E})$ are coaxial, i.e., the stress $\mathbf{T}$ has the same principal axes $\mathbf{N}_i$ as $\mathbf{E}$,

$$\mathbf{T} = \sum_i t_i \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{E} = \sum_i e_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{15}$$

where $t_i$ and $e_i$ are, respectively, the principal stresses and principal strains, and

$$e_i = \frac{1}{2}(\lambda_i^2 - 1) = \frac{1}{2}(\Lambda_i - 1) \tag{16}$$

The second Piola–Kirchhoff stress $\mathbf{T}$, as defined in Eq. (10), is an isotropic function of $\mathbf{C}$. Hence, it has the following representation (Truesdell and Noll, 1965):

$$\mathbf{T} = \varphi_0\mathbf{I} + \varphi_1\mathbf{C} + \varphi_2\mathbf{C}^2 \tag{17}$$

where $\varphi_0$, $\varphi_1$ and $\varphi_2$ are functions of the principal invariants of $\mathbf{C}$. In view of Eq. (12), it implies

$$t_i(\Lambda_1, \Lambda_2, \Lambda_3) = \varphi_0 + \varphi_1\Lambda_i + \varphi_2\Lambda_i^2 \quad i = 1, 2, 3 \tag{18}$$

It is well known that the tensor function $\mathbf{T}$ is differentiable if and only $t_i$ are differentiable (Silhavý, 1997).

By applying the identity

$$\Lambda_j^2 + \Lambda_j\Lambda_i + \Lambda_i^2 - I_1(\Lambda_j + \Lambda_i) + I_2 = 0 (i \neq j)$$

if $\Lambda_i$, $I_1$ and $I_2$ are given, another two eigenvalues $\Lambda_j$ and $\Lambda_k$ yield

$$\Lambda_j, \Lambda_k(\Lambda_i, I_1, I_2) = \frac{1}{2}\left(I_1 - \Lambda_i \pm \sqrt{2I_1\Lambda_i - 3\Lambda_i^2 - 4I_2 + I_1^2}\right) \tag{19}$$

Hence it is convenient to introduce the differentiable scalar response function $f(x_1, x_2, x_3)$ such that

$$f(\Lambda_i, I_1, I_2) = t_1(\Lambda_i, \Lambda_j, \Lambda_k) \tag{20}$$

Similarly, by

$$\Lambda_j^2\Lambda_i + \Lambda_i\Lambda_j^2 - I_1\Lambda_j\Lambda_i + I_3 = 0 \ (i \neq j)$$

we can define the scalar function $\bar{f}(y_1, y_2, y_3)$ such that

$$\bar{f}(\Lambda_i, I_1, I_3) = t_1(\Lambda_i, \Lambda_j, \Lambda_k) \tag{21}$$

In the case of $\Delta_- \neq 0$, Eq. (18) has a unique solution:

$$\left.\begin{array}{lcl} \varphi_0 & = & \frac{1}{\Delta_-}\sum_i f(\Lambda_i, I_1, I_2)\Lambda_j\Lambda_k(\Lambda_k - \Lambda_j) \\ \varphi_1 & = & \frac{1}{\Delta_-}\sum_i f(\Lambda_i, I_1, I_2)(\Lambda_j^2 - \Lambda_k^2) \\ \varphi_2 & = & \frac{1}{\Delta_-}\sum_i f(\Lambda_i, I_1, I_2)(\Lambda_k - \Lambda_j) \end{array}\right\} \tag{22}$$

Here and henceforth, the summation $\sum_i$ is to be carried out for all even permutation $(i, j, k)$ of $(1, 2, 3)$.

Certainly, the scalar coefficients $\varphi_0$, $\varphi_1$ and $\varphi_2$ also can be expressed in terms of

$$J_n = \mathrm{tr}(\mathbf{TC}^n)$$

With the Cayley–Hamilton theorem, the following recursion formula can be established:

$$J_n - I_1 J_{n-1} + I_2 J_{n-2} - I_3 J_{n-3} = 0 \tag{23}$$

In view of (17), we have

$$\begin{aligned} J_0 &= 3\varphi_0 + I_1\varphi_1 + (I_1^2 - 2I_2)\varphi_2 \\ J_{-1} &= \frac{I_2}{I_3}\varphi_0 + 3\varphi_1 + I_1\varphi_2 \\ J_{-2} &= \frac{I_2^2 - 2I_1 I_3}{I_3^2}\varphi_0 + \frac{I_2}{I_3}\varphi_1 + 3\varphi_2 \end{aligned} \tag{24}$$

Rewriting (24) as

$$\mathbf{M}\begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} J_{-2} \\ J_{-1} \\ J_0 \end{pmatrix}$$

where

$$\mathbf{M} = \begin{pmatrix} \dfrac{I_2^2 - 2I_1 I_3}{I_3^2} & \dfrac{I_2}{I_3} & 3 \\ \dfrac{I_2}{I_3} & 3 & I_1 \\ 3 & I_1 & I_1^2 - 2I_2 \end{pmatrix}$$

with

$$\mathbf{M}^{-1} = \frac{1}{\Delta_-^2}\begin{pmatrix} 2(I_1^2 - 3I_2)I_3^2 & (3I_1I_3 - I_1^2 I_2 + 2I_2^2)I_3 & (I_1 I_2 - 9I_3)I_3 \\ (3I_1I_3 - I_1^2 I_2 + 2I_2^2)I_3 & 4I_1I_2I_3 - 2I_1^3 I_3 + I_1^2 I_1^2 - 2I_2^3 - 9I_3^2 & 3I_2I_3 - I_1 I_2^2 + 2I_1^2 I_3 \\ (I_1 I_2 - 9I_3)I_3 & 3I_2I_3 - I_1 I_2^2 + 2I_1^2 I_3 & 2I_2^2 - 6I_1 I_3 \end{pmatrix}$$

then we obtain

$$\begin{aligned} \varphi_0 &= \frac{I_3}{\Delta_-^2}[2I_3(I_1^2 - 3I_2)J_{-2} + (3I_1I_3 - I_1^2 I_2 + 2I_2^2)J_{-1} + (I_1I_2 - 9I_3)J_0] \\ \varphi_1 &= \frac{1}{\Delta_-^2}[I_3(3I_1I_3 - I_1^2 I_2 + 2I_2^2)J_{-2} + (4I_1I_2I_3 - 2I_1^3 I_3 + I_1^2 I_1^2 - 2I_2^3 - 9I_3^2)J_{-1} + (3I_2I_3 - I_1 I_2^2 + 2I_1^2 I_3)J_0] \\ \varphi_2 &= \frac{1}{\Delta_-^2}[I_3(I_1 I_2 - 9\mathbf{I}_3)J_{-2} + (3I_2I_3 - I_1 I_2^2 + 2I_1^2 I_3)J_{-1} + (2I_2^2 - 6I_1I_3)J_0] \end{aligned} \tag{25}$$

In the case that **C** has only two distinct eigenvalues, without loss of generality, by assuming $\Lambda_1(t) \neq \Lambda_2(t) = \Lambda_3(t) = \Lambda_0(t)$ and using the relations

$$\mathbf{C} = \Lambda_2\mathbf{I} + (\Lambda_1 - \Lambda_2)\mathbf{N}_1 \otimes \mathbf{N}_1 \tag{26}$$

and

$$\mathbf{C}^2 = (\Lambda_1 + \Lambda_2)\mathbf{C} - \Lambda_1\Lambda_2\mathbf{I} \tag{27}$$

**T** has the reduced representation

$$\mathbf{T} = \bar{\varphi}_0 \mathbf{I} + \bar{\varphi}_1 \mathbf{C} \tag{28}$$

which implies

$$\bar{\varphi}_0 + \bar{\varphi}_1 \Lambda_i = f(\Lambda_i, I_1, I_2) \quad i = 1, 2 \tag{29}$$

The solution of Eq. (29) is given by

$$\begin{aligned} \bar{\varphi}_0 &= \frac{\Lambda_2 f(\Lambda_1, I_1, I_2) - \Lambda_1 f(\Lambda_2, I_1, I_2)}{\Lambda_2 - \Lambda_1} \\ \bar{\varphi}_1 &= \frac{f(\Lambda_2, I_1, I_2) - f(\Lambda_1, I_1, I_2)}{\Lambda_2 - \Lambda_1} \end{aligned} \tag{30}$$

Analogous analysis for the distinct case, Eq. (23) may reduce to

$$\begin{aligned} J_0 &= 3\bar{\varphi}_0 + I_1 \bar{\varphi}_1 \\ J_{-1} &= \frac{I_2}{I_3} \bar{\varphi}_0 + 3\bar{\varphi}_1 \end{aligned} \tag{31}$$

so that

$$\begin{aligned} \bar{\varphi}_0 &= \frac{I_3}{I_1 I_2 - 9I_3} (I_1 J_{-1} - 3J_0) \\ \bar{\varphi}_1 &= \frac{1}{I_1 I_2 - 9I_3} (I_2 J_0 - 3I_3 J_{-1}) \end{aligned} \tag{32}$$

It is interesting that, by choosing $\mathbf{T} = \mathbf{C}^2$, the comparison of (27) with (28) gives

$$\begin{aligned} \Lambda_1 \Lambda_2 &= \frac{I_3}{I_1 I_2 - 9I_3} (2I_1^2 - 6I_2) \\ \Lambda_1 + \Lambda_2 &= \frac{1}{I_1 I_2 - 9I_3} (I_1^2 I_2 - 2I_2^2 - 3I_1 I_3) \end{aligned} \tag{33}$$

In the case that three eigenvalues of **C** coalesce, if assuming $\Lambda_1(t) = \Lambda_2(t) = \Lambda_3(t) = \Lambda(t)$ and $t_1 = t_2 = t_3 = \hat{\mathrm{f}}(\Lambda)$, that is, the deformation corresponds to a state of pure dilation, then **C** and **T** has the following representation

$$\mathbf{C} = \Lambda \mathbf{I} \tag{34}$$

and

$$\mathbf{T} = \hat{f}(\Lambda) \mathbf{I} \tag{35}$$

where $\hat{f}(\Lambda) = t_1(\Lambda, \Lambda, \Lambda)$.

## 4. Basis-free expressions for the time rates of stress tensor

### 4.1. Case of distinct eigenvalues

If we decompose $\dot{\mathbf{T}}$ and $\dot{\mathbf{E}}$ in the principal axes $\mathbf{N}_i$ and denote their components by $\dot{T}_{ij}$ and $\dot{E}_{ij}$, respectively, then we have

$$\dot{\mathbf{E}} = \sum_{i,j} \dot{E}_{ij} \mathbf{N}_i \otimes \mathbf{N}_j \tag{36}$$

and

$$\dot{\mathbf{T}} = \sum_{i,j} \dot{T}_{ij}\mathbf{N}_i \otimes \mathbf{N}_j \tag{37}$$

respectively, where

$$\dot{E}_{ij} = \mathbf{N}_i \cdot \dot{\mathbf{E}}\mathbf{N}_j, \quad \dot{T}_{ij} = \mathbf{N}_i \cdot \dot{\mathbf{T}}\mathbf{N}_j$$

The time rates are often advantageously obtained by the principal method (Hill, 1978). If $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3 \neq \Lambda_1$, the components of (37) can be expressed in terms of $\dot{E}_{ij}$ as (Ogden, 1974)

$$\dot{T}_{ij} = \begin{cases} \sum_k \frac{\partial t_i}{\partial \Lambda_k}\dot{\Lambda}_k & \text{if } i = j \\ 2\frac{t_i - t_j}{\Lambda_i - \Lambda_j}\dot{E}_{ij} & \text{if } i \neq j \end{cases} \tag{38}$$

However, further abstract or numerical manipulations with the principal method would require not only the eigenvalues but also the eigenvectors of the stretch tensor to be determined.

Next, we set about representing the stress rate in basis-free form. Substituting (18) and (38) into (37), and comparing the components of $\dot{\mathbf{E}}$ in (36), we have

$$\dot{\mathbf{T}} = \sum_{i,j=1}^{3} \frac{\partial t_i}{\partial \Lambda_j}\dot{\Lambda}_j\mathbf{N}_i \otimes \mathbf{N}_i - \sum_{i=1}^{3}(\varphi_1 + 2\varphi_2\Lambda_i)\dot{\Lambda}_i N_i \otimes \mathbf{N}_i + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) \tag{39}$$

The identities

$$\Lambda_i^3 - I_1\Lambda_i^2 + I_2\Lambda_i - I_3 = 0 \tag{40}$$

leads to

$$\dot{\Lambda}_i = (3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(\Lambda_i^2\dot{I}_1 - \Lambda_i\dot{I}_2 + \dot{I}_3) \tag{41}$$

Therefore

$$\begin{aligned} \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial t_i}{\partial \Lambda_j}\dot{\Lambda}_j\mathbf{N}_i \otimes \mathbf{N}_i = {} & \left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2 - 2I_1\Lambda_j + I_2)^{-1}\Lambda_j^2\right]\mathbf{N}_i \otimes \mathbf{N}_i\right\}\dot{I}_1 \\ & - \left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2 - 2I_1\Lambda_j + I_2)^{-1}\Lambda_j\right]\mathbf{N}_i \otimes \mathbf{N}_i\right\}\dot{I}_2 \\ & + \left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2 - 2I_1\Lambda_j + I_2)^{-1}\right]\mathbf{N}_i \otimes \mathbf{N}_i\right\}\dot{I}_3 \end{aligned} \tag{42}$$

Since

$$\begin{aligned} \dot{I}_1 &= 2\,\mathrm{tr}\,\dot{\mathbf{E}}, \\ \dot{I}_2 &= 2I_1\,\mathrm{tr}\,\dot{\mathbf{E}} - 2\,\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}} \\ \dot{I}_3 &= 2I_2\,\mathrm{tr}\,\dot{\mathbf{E}} - 2I_1\,\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}} + 2\,\mathrm{tr}\,\mathbf{C}^2\dot{\mathbf{E}} \end{aligned} \tag{43}$$

thus

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}\dot{\Lambda}_j\mathbf{N}_i\otimes\mathbf{N}_i = 2\left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}(I_2-2I_1\Lambda_j+\Lambda_j^2)\right]\mathbf{N}_i\otimes\mathbf{N}_i\right\}\mathrm{tr}\,\dot{\mathbf{E}}$$
$$-2\left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}(I_1-\Lambda_j)\right]\mathbf{N}_i\otimes\mathbf{N}_i\right\}\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}}$$
$$+2\left\{\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}\right]\mathbf{N}_i\otimes\mathbf{N}_i\right\}\mathrm{tr}\,\mathbf{C}^2\dot{\mathbf{E}} \tag{44}$$

Denote

$$\mathbf{H}_1(\mathbf{C}) = 2\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}(I_2-2I_1\Lambda_j+\Lambda_j^2)\right.$$
$$\left.-(\varphi_1+2\varphi_2\Lambda_i)(3\Lambda_i^2-2I_1\Lambda_i+I_2)^{-1}(I_2-2I_1\Lambda_i+\Lambda_i^2)\right]\mathbf{N}_i\otimes\mathbf{N}_i$$

$$\mathbf{H}_2(\mathbf{C}) = -2\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}(I_1-\Lambda_j)\right. \tag{45}$$
$$\left.-(\varphi_1+2\varphi_2\Lambda_i)(3\Lambda_i^2-2I_1\Lambda_i+I_2)^{-1}(I_1-\Lambda_i)\right]\mathbf{N}_i\otimes\mathbf{N}_i$$

$$\mathbf{H}_3(\mathbf{C}) = 2\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial t_i}{\partial \Lambda_j}(3\Lambda_j^2-2I_1\Lambda_j+I_2)^{-1}-(\varphi_1+2\varphi_2\Lambda_i)(3\Lambda_i^2-2I_1\Lambda_i+I_2)^{-1}\right]\mathbf{N}_i\otimes\mathbf{N}_i$$

Thus the time rate of $\mathbf{T}$ can be written as

$$\dot{\mathbf{T}} = \mathbf{H}_1(\mathbf{C})\,\mathrm{tr}\,\dot{\mathbf{E}} + \mathbf{H}_2(\mathbf{C})\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}}) + \mathbf{H}_3(\mathbf{C})\mathrm{tr}(\mathbf{C}^2\dot{\mathbf{E}}) + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}}+\dot{\mathbf{E}}\mathbf{C}) \tag{46}$$

Clearly, the tensor functions $\mathbf{H}_j = \mathbf{H}_j(\mathbf{C})$ are isotropic functions of $\mathbf{C}$ and have the representations

$$\mathbf{H}_j(\mathbf{C}) = \alpha_{1j}\mathbf{I} + \alpha_{2j}\mathbf{C} + \alpha_{3j}\mathbf{C}^2 \tag{47}$$

Hence the representation (46) could be written in the standard form (Rivlin and Ericksen, 1955)

$$\dot{\mathbf{T}} = \alpha_1\mathbf{I} + \alpha_2\mathbf{C} + \alpha_3\mathbf{C}^2 + \alpha_4\dot{\mathbf{E}} + \alpha_5(\mathbf{C}\dot{\mathbf{E}}+\dot{\mathbf{E}}\mathbf{C}) + \alpha_6(\mathbf{C}^2\dot{\mathbf{E}}+\dot{\mathbf{E}}\mathbf{C}^2) \tag{48}$$

where

$$\alpha_i = \alpha_{i1}\mathrm{tr}(\dot{\mathbf{E}}) + \alpha_{i2}\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}}) + \alpha_{i3}\,\mathrm{tr}(\mathbf{C}^2\dot{\mathbf{E}}) \quad (i=1,2,3)$$

By (47), it can be determined that the coefficients $\alpha_{ij}(i,j=1,2,3)$ and $\alpha_k$ ($k$ = 4, 5, 6) in (48) are as follows:

$$\alpha_4 = 2\varphi_1, \quad \alpha_5 = 2\varphi_2, \quad \alpha_6 = 0$$

and

$$\alpha_{11} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_2 - 2I_1\Lambda_l + \Lambda_l^2) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_2 - 2I_1\Lambda_i + \Lambda_i^2)\right]\Lambda_j\Lambda_k(\Lambda_k - \Lambda_j)$$

$$\alpha_{21} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_2 - 2I_1\Lambda_l + \Lambda_l^2) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_2 - 2I_1\Lambda_l + \Lambda_i^2)\right](\Lambda_j^2 - \Lambda_k^2)$$

$$\alpha_{31} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_2 - 2I_1\Lambda_l + \Lambda_l^2) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_2 - 2I_1\Lambda_i + \Lambda_i^2)\right](\Lambda_k - \Lambda_j)$$

$$\alpha_{12} = -\frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_1 - \Lambda_l) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_1 - \Lambda_i)\right]\Lambda_j\Lambda_k(\Lambda_k - \Lambda_j) \qquad (49)$$

$$\alpha_{22} = -\frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_1 - \Lambda_l) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_1 - \Lambda_i)\right](\Lambda_j^2 - \Lambda_k^2)$$

$$\alpha_{32} = -\frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1}(I_1 - \Lambda_l) - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}(I_2 - \Lambda_i)\right](\Lambda_j - \Lambda_k)$$

$$\alpha_{13} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1} - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}\right]\Lambda_j\Lambda_k(\Lambda_k - \Lambda_j)$$

$$\alpha_{23} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1} - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}\right](\Lambda_j^2 - \Lambda_k^2)$$

$$\alpha_{33} = \frac{2}{\Delta_-}\sum_i \left[\sum_{l=1}^{3}\frac{\partial t_i}{\partial \Lambda_l}(3\Lambda_l^2 - 2I_1\Lambda_l + I_2)^{-1} - (\varphi_1 + 2\varphi_2\Lambda_i)(3\Lambda_i^2 - 2I_1\Lambda_i + I_2)^{-1}\right](\Lambda_j - \Lambda_k)$$

In particular, for the Green elastic material, due to the property $\alpha_{ij} = \alpha_{ji}$ the expression (48) reduces to

$$\dot{\mathbf{T}} = \alpha_{11}\mathbf{I}\,\mathrm{tr}\,\dot{\mathbf{E}} + \alpha_{12}(\mathbf{I}\,\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}} + \mathbf{C}\,\mathrm{tr}\,\dot{\mathbf{E}}) + \alpha_{13}(\mathbf{I}\,\mathrm{tr}\,\mathbf{C}^2\dot{\mathbf{E}} + \mathbf{C}^2\,\mathrm{tr}\,\dot{\mathbf{E}}) + \alpha_{22}\mathbf{C}\,\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}} + \alpha_{23}(\mathbf{C}\,\mathrm{tr}\,\mathbf{C}^2\dot{\mathbf{E}} + \mathbf{C}^2\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}}) + \alpha_{33}\mathbf{C}^2\mathrm{tr}\,\mathbf{C}^2\dot{\mathbf{E}}\alpha_4\dot{\mathbf{E}} + \alpha_5(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) + \alpha_6(\mathbf{C}^2\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}^2) \qquad (48')$$

On the other hand, the representation for the stress rate can be written in the different forms in terms of the scalar function $f$.

Denote

$$\mathbf{G}_j = \sum_i f_{,j}(\Lambda_i, I_1, I_2)\mathbf{N}_i \otimes \mathbf{N}_i, \quad \overline{\mathbf{G}}_j = \sum_i \bar{f}_{,j}(\Lambda_i, I_1, I_3)\mathbf{N}_i \otimes \mathbf{N}_i \tag{50}$$

where $f_{,i} = \frac{\partial f}{\partial x_i}(x_1, x_2, x_3)$ and $\bar{f}_{,i} = \frac{\partial \bar{f}}{\partial y_i}(y_1, y_2, y_3)$. Obviously, the tensor functions $\mathbf{G}_j = \mathbf{G}_j(\mathbf{C})$ and $\overline{\mathbf{G}}_j = \overline{\mathbf{G}}_j(\mathbf{C})$ are isotropic functions of $\mathbf{C}$.

Thus, we have

$$\sum_{i,j=1}^{3} \frac{\partial t_i}{\partial \Lambda_j}\dot{\Lambda}_j \mathbf{N}_i \otimes \mathbf{N}_i = \mathbf{G}_1(\mathbf{C})\sum_i \dot{\Lambda}_i\ \mathbf{N}_i \otimes \mathbf{N}_i + \mathrm{tr}(\dot{\mathbf{C}})(\mathbf{G}_2(\mathbf{C}) + I_1\mathbf{G}_3(\mathbf{C})) - \mathrm{tr}(\mathbf{C}\dot{\mathbf{C}})\mathbf{G}_3(\mathbf{C}) \tag{51}$$

or

$$\sum_{i,j=1}^{3} \frac{\partial t_i}{\partial \Lambda_j}\dot{\Lambda}_j \mathbf{N}_i \otimes \mathbf{N}_i = \overline{\mathbf{G}}_1(\mathbf{C})\sum_i \dot{\Lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i + \mathrm{tr}(\dot{\mathbf{C}})\overline{\mathbf{G}}_2(\mathbf{C}) + I_3\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{C}})\overline{\mathbf{G}}_3(\mathbf{C}) \tag{52}$$

To substitute (51) and (52) into (39), the time rate of $\mathbf{T}$ can be written as

$$\begin{aligned}\dot{\mathbf{T}} = {} & (\mathbf{G}_1(\mathbf{C}) - \varphi_1\mathbf{I} - 2\varphi_2\mathbf{C})\left(\sum_{i=1}^{3} \dot{\Lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i\right) + 2\mathrm{tr}(\dot{\mathbf{E}})(\mathbf{G}_2(\mathbf{C}) + I_1\mathbf{G}_3(\mathbf{C})) - 2\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})\mathbf{G}_3(\mathbf{C}) \\ & + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C})\end{aligned} \tag{53}$$

or

$$\begin{aligned}\dot{\mathbf{T}} = {} & (\overline{\mathbf{G}}_1(\mathbf{C}) - \varphi_1\mathbf{I} - 2\varphi_2\mathbf{C})\left(\sum_{i=1}^{3} \dot{\Lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i\right) + 2\,\mathrm{tr}(\dot{\mathbf{E}})\overline{\mathbf{G}}_2(\mathbf{C}) + 2I_3\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\overline{\mathbf{G}}_3(\mathbf{C}) + 2\varphi_1\dot{\mathbf{E}} \\ & + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C})\end{aligned} \tag{54}$$

Furthermore, since the solution of the tensor equation

$$\mathbf{C}\mathbf{X} - \mathbf{X}\mathbf{C} = \dot{\mathbf{C}} - \sum_i \dot{\Lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{55}$$

is (Dui, 1999)

$$\mathbf{X} = \Delta_{-}^{-2}\widehat{\mathbf{C}}(\mathbf{C}\dot{\mathbf{C}} - \mathbf{C}\dot{\mathbf{C}})\widehat{\mathbf{C}} \tag{56}$$

substituting (55) into (56) yields

$$\sum_i \dot{\Lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i = \dot{\mathbf{C}} - \Delta_{-}^{-2}\hat{\mathbf{C}}(\mathbf{C}^2\dot{\mathbf{C}} - 2\mathbf{C}\dot{\mathbf{C}}\mathbf{C} + \dot{\mathbf{C}}\mathbf{C}^2)\widehat{\mathbf{C}} \tag{57}$$

In view of the explicit expression in (57), the stress rate is

$$\begin{aligned}\dot{\mathbf{T}} = {} & 2(\mathbf{G}_1 - \varphi_1\mathbf{I} - 2\varphi_2\mathbf{C})[\dot{\mathbf{E}} - \Delta_{-}^{-2}\hat{\mathbf{C}}(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2)\hat{\mathbf{C}}] + 2\ \mathrm{tr}(\dot{\mathbf{E}})(\mathbf{G}_2 + I_1\mathbf{G}_3) - 2\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})\mathbf{G}_3 \\ & + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C})\end{aligned} \tag{58}$$

or

$$\dot{\mathbf{T}} = 2(\overline{\mathbf{G}}_1 - \varphi_1\mathbf{I} - 2\varphi_2\mathbf{C})[\dot{\mathbf{E}} - \Delta_-^{-2}\widehat{\mathbf{C}}(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2)\hat{\mathbf{C}}] + 2\mathrm{tr}(\dot{\mathbf{E}})\overline{\mathbf{G}}_2 + 2I_3\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\overline{\mathbf{G}}_3 + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) \tag{59}$$

It is seen that expressions (58) and (59) are more concise than (48).

**Remark.** If the coefficients are differentiable, to differentiate the stress (17) directly with respect to time yields (Itskov, 2000)

$$\dot{\mathbf{T}} = \dot{\varphi}_0\mathbf{I} + \dot{\varphi}_1\mathbf{C} + \dot{\varphi}_2\mathbf{C}^2 + 2\varphi_1\dot{\mathbf{E}} + 2\varphi_2(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) \tag{60}$$

But we should be aware of that the coefficients $\varphi_i$ may not be differentiable even if $\mathbf{T}$ is differentiable (Ball, 1984 and Man, 1995). Hence the formula (60) must be regarded as incomplete ones.

### 4.2. Double coalescence

In the case that $\mathbf{C}$ has only two distinct eigenvalues, i.e. $\Lambda_1(t) \neq \Lambda_2(t) = \Lambda_3(t) = \Lambda_0(t)$ the components of $\dot{\mathbf{E}}$ and $\dot{\mathbf{T}}$ are such that

$$\dot{E}_{23} = 0, \quad \dot{T}_{22} = \dot{T}_{33}, \quad \dot{T}_{23} = 0 \tag{61}$$

and (Chadwick and Ogden, 1971)

$$\left.\begin{aligned} &\dot{T}_{11} = \sum_j \frac{\partial t_1}{\partial \Lambda_j}\dot{\Lambda}_j, \quad \dot{T}_{22} = \sum_j \frac{\partial t_2}{\partial \Lambda_j}\dot{\Lambda}_j \\ &\dot{T}_{12} = 2\frac{t_1 - t_2}{\Lambda_1 - \Lambda_2}\dot{E}_{12} \end{aligned}\right\} \tag{62}$$

**Remark.** Gurtin and Spear (1983) and Hoger (1986) have mentioned that the early derivations of formulae (62) is not rigorous, the rigorous proof should follow Scheidler (1991).

Comparing the components of (62), we have

$$\dot{\mathbf{T}} = \left(\sum_j \frac{\partial t_2}{\partial \Lambda_j}\dot{\Lambda}_j - \bar{\varphi}_1\dot{\Lambda}_2\right)\mathbf{I} + \left[\left(\sum_j \frac{\partial t}{\partial \Lambda_j}\dot{\Lambda}_j - \bar{\varphi}_1\dot{\Lambda}_1\right) - \left(\sum_j \frac{\partial t_2}{\partial \Lambda_j}\dot{\Lambda}_j - \bar{\varphi}_1\dot{\Lambda}_2\right)\right]\mathbf{N}_1 \otimes \mathbf{N}_1 + 2\bar{\varphi}_1\dot{\mathbf{E}} \tag{63}$$

In this case, $\mathbf{T}$ has the reduced representation

$$\dot{\mathbf{T}} = \bar{\alpha}_1\mathbf{I} + \bar{\alpha}_2\mathbf{C} + \bar{\alpha}_4\dot{\mathbf{E}} + \bar{\alpha}_5(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) \tag{64}$$

where

$$\bar{\alpha}_1 = \bar{\alpha}_{11}\,\mathrm{tr}(\dot{\mathbf{E}}) + \bar{\alpha}_{12}\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})$$
$$\bar{\alpha}_2 = \bar{\alpha}_{21}\,\mathrm{tr}(\dot{\mathbf{E}}) + \bar{\alpha}_{22}\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})$$

To compare (63) with (64), the coefficients $\bar{\alpha}_{\alpha\beta}(\alpha,\ \beta = 1, 2)$ and $\bar{\alpha}_k$ $(k = 4, 5)$ in (64) can be presented as follows:

$$\bar{\alpha}_4 = 2\bar{\varphi}_1, \quad \bar{\alpha}_5 = 0$$

and

$$\begin{aligned}
\bar{\alpha}_{11} &= \frac{2}{\overline{\Delta}^2}\left(\frac{\partial_1}{\partial \Lambda_1}\Lambda_2^2 + \frac{\partial t_2}{\partial \Lambda_2}\Lambda_1^2 - \frac{\partial t_2}{\partial \Lambda_1}\overline{I}_2 - \frac{\partial t_1}{\partial \Lambda_2}\overline{I}_2 - \frac{1}{2}(I_1^2 - I_2)\bar{\varphi}_1\right)\\
\bar{\alpha}_{12} &= \frac{-2}{\overline{\Delta}^2}\left(\frac{\partial t_1}{\partial \Lambda_1}\Lambda_2 + \frac{\partial t_2}{\partial \Lambda_2}\Lambda_1 - \frac{\partial t_2}{\partial \Lambda_1}\Lambda_1 - \frac{\partial t_1}{\partial \Lambda_2}\Lambda_2 - \frac{1}{2}I_1\bar{\varphi}_1\right)\\
\bar{\alpha}_{21} &= \frac{-2}{\overline{\Delta}^2}\left(\frac{\partial t_1}{\partial \Lambda_1}\Lambda_2 + \frac{\partial t_2}{\partial \Lambda_2}\Lambda_1 - \frac{\partial t_2}{\partial \Lambda_1}\Lambda_2 - \frac{\partial t_1}{\partial \Lambda_2}\Lambda_1 - \frac{1}{2}I_1\bar{\varphi}_1\right)\\
\bar{\alpha}_{22} &= \frac{1}{\overline{\Delta}^2}\left(\frac{\partial t_1}{\partial \Lambda_1} + \frac{\partial t_2}{\partial \Lambda_2} - \frac{\partial t_2}{\partial \Lambda_1} - \frac{\partial t_1}{\partial \Lambda_2} - \frac{3}{2}\bar{\varphi}_1\right)
\end{aligned} \tag{65}$$

where $\overline{\Delta}^2 = I_1^2 - 4I_2$.

Next, we give the representation for the stress rate in terms of the scalar function $f$.

Since $\Lambda_2 = \Lambda_3 = \frac{1}{2}(I_1 - \Lambda_1)$, we define

$$f_{,1}(\Lambda_1, I_1, I_2) = \frac{\partial t_1}{\partial \Lambda_1}(\Lambda_1, \Lambda_2, \Lambda_3) - \frac{1}{2}\left[\frac{\partial t_1}{\partial \Lambda_2}(\Lambda_1, \Lambda_2, \Lambda_3) + \frac{\partial t_1}{\partial \Lambda_3}(\Lambda_1, \Lambda_2, \Lambda_3)\right]$$

$$f_{,2}(\Lambda_1, I_1, I_2) = \frac{1}{2}\left[\frac{\partial t_1}{\partial \Lambda_2}(\Lambda_1, \Lambda_2, \Lambda_3) + \frac{\partial t_1}{\partial \Lambda_3}(\Lambda_1, \Lambda_2, \Lambda_3)\right]$$

$$f_{,3}(\Lambda_1, I_1, I_2) = 0$$

For $i = 2$, 3, we may use the same definition of $f_{,j}(\Lambda_{\rm i}, I_1, I_2)$ as that in the last section.

Denote

$$\mathbf{G}_j = f_{,j}(\Lambda_2, I_1, I_2)\mathbf{I} + (f_{,j}(\Lambda_1, I_1, I_2) - f_{,j}(\Lambda_2, I_1, I_2))\mathbf{N}_1 \otimes \mathbf{N}_1 \tag{66}$$

Thus (63) can be written as

$$\dot{\mathbf{T}} = (\mathbf{G}_1(\mathbf{C}) - \bar{\varphi}_1\mathbf{I})(\dot{\Lambda}_2\mathbf{I} + (\dot{\Lambda}_1 - \dot{\Lambda}_2)\mathbf{N}_1 \otimes \mathbf{N}_1) + 2\,\mathrm{tr}(\dot{\mathbf{E}})(\mathbf{G}_2(\mathbf{C}) + I_1\mathbf{G}_3(\mathbf{C})) - 2\,\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})\mathbf{G}_3(\mathbf{C}) + 2\bar{\varphi}_1\dot{\mathbf{E}} \tag{67}$$

or

$$\dot{\mathbf{T}} = (\overline{\mathbf{G}}_1(\mathbf{C}) - \bar{\varphi}_1\mathbf{I})(\dot{\Lambda}_2\mathbf{I} + (\dot{\Lambda}_1 - \dot{\Lambda}_2)\mathbf{N}_1 \otimes \mathbf{N}_1) + 2\,\mathrm{tr}(\dot{\mathbf{E}})\overline{\mathbf{G}}_2(\mathbf{C}) + 2I_3\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\overline{\mathbf{G}}_3(\mathbf{C}) + 2\bar{\varphi}_1\dot{\mathbf{E}} \tag{68}$$

Since the solution of the tensor equation

$$\mathbf{CX} - \mathbf{XC} = \dot{\mathbf{C}} - \dot{\Lambda}_2\mathbf{I} - (\dot{\Lambda}_1 - \dot{\Lambda}_2)\mathbf{N}_1 \otimes \mathbf{N}_1 \tag{69}$$

is (Guo et al., 1992)

$$\mathbf{X} = (I_1^2 - 4I_2)^{-1}(\mathbf{C}\dot{\mathbf{C}} - \dot{\mathbf{C}}\mathbf{C}) \tag{70}$$

we have

$$\dot{\Lambda}_2\mathbf{I} + (\dot{\Lambda}_1 - \dot{\Lambda}_2)\mathbf{N}_1 \otimes \mathbf{N}_1 = \dot{\mathbf{C}} - (I_1^2 - 4I_2)^{-1}(\mathbf{C}^2\dot{\mathbf{C}} - 2\mathbf{C}\dot{\mathbf{C}}\mathbf{C} + \dot{\mathbf{C}}\mathbf{C}^2) \tag{71}$$

By the identity (71), the time rate of stress yields

$$\begin{aligned}\dot{\mathbf{T}} = {} & 2(\mathbf{G}_1(\mathbf{C}) - \bar{\varphi}_1\mathbf{I})(\dot{\mathbf{E}}) - (I_1^2 - 4I_2)^{-1}(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2) + 2\,\mathrm{tr}(\dot{\mathbf{E}})(\mathbf{G}_2(\mathbf{C}) + I_1\mathbf{G}_3(\mathbf{C}))\\ & - 2\mathrm{tr}(\mathbf{C}\dot{\mathbf{E}})\mathbf{G}_3(\mathbf{C}) + 2\bar{\varphi}_1\dot{\mathbf{E}}\end{aligned} \tag{72}$$

or

$$\dot{\mathbf{T}} = 2(\overline{\mathbf{G}}_1(\mathbf{C}) - \bar{\varphi}_1\mathbf{I})(\dot{\mathbf{E}}) - (I_1^2 - 4I_2)^{-1}(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2) + 2\,\mathrm{tr}(\dot{\mathbf{E}})\overline{\mathbf{G}}_2(\mathbf{C}) + 2I_3\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\overline{\mathbf{G}}_3(\mathbf{C}) + 2\bar{\varphi}_1\dot{\mathbf{E}} \tag{73}$$

### 4.3. Triple coalescence

In the case that three eigenvalues of $\mathbf{C}$ coalesce, we have

$$\dot{\mathbf{T}} = 2\widehat{\mathbf{G}}(\mathbf{C})\dot{\mathbf{E}} \tag{74}$$

where

$$\widehat{\mathbf{G}}(\mathbf{C}) = \hat{f}'(\Lambda)\mathbf{I}$$

$$\hat{f}'(\Lambda) = \frac{\partial t_1}{\partial \Lambda_1}(\Lambda_1,\Lambda_2,\Lambda_3) + \frac{\partial t_1}{\partial \Lambda_2}(\Lambda_1,\Lambda_2,\Lambda_3) + \frac{\partial t_1}{\partial \Lambda_3}(\Lambda_1,\Lambda_2,\Lambda_3)\bigg|_{\Lambda_1=\Lambda_2,=\Lambda_3=\Lambda}$$

It should be noted that present derivations of the basis-free representations for the stress rate do not involve the time rate of coefficients and principal directions of the stress tensor.

## 5. Examples

In the previous section, some basis-free expressions for the stress rate $\dot{\mathbf{T}}$ are obtained. The results are valid in the cases that the eigenvalues of $\mathbf{C}$ are distinct, doubly coalescent, and triply coalescent, respectively. Let us calculate some examples for illustration.

### 5.1. Example 1

Consider the stress tensor as

$$\mathbf{T}(\mathbf{C}) = I_1 I_3 \mathbf{C}^{-1} \tag{75}$$

In view of the Cayley–Hamilton theorem, it has the representation in the form of (17)

$$\mathbf{T}(\mathbf{C}) = I_1 I_2 \mathbf{I} - I_1^2\mathbf{C} + I_1\mathbf{C}^2 \tag{76}$$

and

$$\varphi_0 = I_1 I_2, \quad \varphi_1 = -I_1^2, \quad \varphi_2 = I_1$$

By use of the spectral theorem, (76) admits

$$\mathbf{T}(\mathbf{C}) = \sum_i I_1(\Lambda_i^2 - I_1\Lambda_i + I_2)\mathbf{N}_i \otimes \mathbf{N}_i = \sum_i I_1 I_3 \Lambda_i^{-1}\mathbf{N}_i \otimes \mathbf{N}_i \tag{77}$$

Thus the response function $f$ may be found

$$f(\Lambda_i, I_1, I_2) = I_1(\Lambda_i^2 - I_1\Lambda_i + I_2) \tag{78}$$

The related functions are

$$f_{,1}(\Lambda_i, I_1, I_2) = 2I_1\Lambda_i - I_1^2, \quad f_{,2}(\Lambda_i, I_1, I_2) = \Lambda_i^2 - 2I_1\Lambda_i + I_2, \quad f_{,3}(\Lambda_i, I_1, I_2) = I_1$$

and

$$\mathbf{G}_1(\mathbf{C}) = 2I_1\mathbf{C} - I_1^2\mathbf{I}, \quad \mathbf{G}_2(\mathbf{C}) = \mathbf{C}^2 - 2I_1C + I_2\mathbf{I}, \quad \mathbf{G}_3(\mathbf{C}) = I_1\mathbf{I}$$

In terms of formula (58), the stress rate is given by

$$\dot{\mathbf{T}} = 2\,\mathrm{tr}(\dot{\mathbf{E}})(\mathbf{C}^2 - 2I_1\mathbf{C} + (I_2 + I_1^2)\mathbf{I}) - 2I_1(\mathrm{tr}\,\mathbf{C}\dot{\mathbf{E}})\mathbf{I} - 2I_1^2\dot{\mathbf{E}} + 2I_1(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) \tag{79}$$

Similarly, the function $\bar{f}$ is

$$\bar{f}(\Lambda_i, I_1, I_3) = I_1 I_3 \Lambda_i^{-1}$$

and

$$\bar{f}_{,1}(\Lambda_i, I_1, I_3) = -I_1 I_3 \Lambda_i^{-2}, \quad \bar{f}_{,2}(\Lambda_i, I_1, I_3) = I_3\Lambda_i^{-1}, \quad \bar{f}_{,3}(\Lambda_i, I_1, I_3) = I_1\Lambda_i^{-1}$$

$$\overline{\mathbf{G}}_1(\mathbf{C}) = -I_1 I_3\mathbf{C}^{-2}, \quad \overline{\mathbf{G}}_2(\mathbf{C}) = I_3\mathbf{C}^{-1}, \quad \overline{\mathbf{G}}_3(\mathbf{C}) = I_1\mathbf{C}^{-1}$$

By (59), the stress rate can be given by

$$\dot{\mathbf{T}} = 2I_3((I_1\,\mathrm{tr}\,\mathbf{C}^{-1}\dot{\mathbf{E}} + \mathrm{tr}\dot{\mathbf{E}})\mathbf{C}^{-1} - I_1\mathbf{C}^{-1}\dot{\mathbf{E}}\mathbf{C}^{-1}) \tag{80}$$

To express (79) in the form (48), we have

$$\alpha_{11} = 2(I_1^2 + I_2), \quad \alpha_{12} = -2I_1, \quad \alpha_{21} = -4I_1, \quad \alpha_{31} = 2, \quad \alpha_4 = -2I_1^2, \quad \alpha_5 = 2I_1$$

and others are 0. Since $\alpha_{12} \neq \alpha_{21}$ and $\alpha_{13} \neq \alpha_{31}$, it demonstrates that the formula (48′) for the Green elastic material is not valid for the general isotropic one.

### 5.2. Example 2

Consider a frequently applied strain energy of Ogden material model

$$w = \sum_{p=1}^{N} \mu_p\left(\Lambda_1^{\alpha_p} + \Lambda_2^{\alpha_p} + \Lambda_3^{\alpha_p} - 3I_3^{\alpha_p/3/\alpha_p}\right)\Big/\alpha_p + h(I_3) \tag{81}$$

The second Piola–Kirchhoff stress tensor is

$$\mathbf{T} = \sum_{p=1}^{N} \mu_p\left(\mathbf{C}^{\alpha_p-1} - I_3^{\alpha_p/(3-1)} I_3\mathbf{C}^{-1}\right) + h'(I_3)I_3\mathbf{C}^{-1} \tag{82}$$

(A) In the case that $\mathbf{C}$ has three distinct eigenvalues
From (82), we have

$$\bar{f}(\lambda_i, I_1, I_3) = \sum_{p=1}^{N} \mu_p\left(\lambda_i^{\alpha_p-1} - I_3^{\alpha_p/3}\lambda_i^{-1}\right) + h'(I_3)I_3\lambda_i^{-1} \tag{83}$$

By use of the representation (17), we know

$$\mathbf{C}^{\alpha_p-1} = \varphi_{0p}\mathbf{I} + \varphi_{1p}\mathbf{C} + \varphi_{2p}\mathbf{C}^2$$

where $\varphi_{ip}$ can be determined by (83) and (22).

Hence, the time rate of the stress can be obtained by (59)

$$\dot{\mathbf{T}} = \sum_{p=1}^{N} \mu_p \{[(\alpha_p - 1)\mathbf{C}^{\alpha_p-2} - \varphi_{1p}\mathbf{I} - 2\varphi_{2p}\mathbf{C}][\dot{\mathbf{E}} - \Delta_-^{-2}\hat{\mathbf{C}}(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2)\hat{\mathbf{C}}] + 2\varphi_{1p}\dot{\mathbf{E}}$$
$$+ 2\varphi_{2p}(\mathbf{C}\dot{\mathbf{E}} + \dot{\mathbf{E}}\mathbf{C}) - \frac{2\alpha_p}{3} I_3^{\alpha_p/3}\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\mathbf{C}^{-1} + 2I_3^{\alpha_p/3}\mathbf{C}^{-1}\dot{\mathbf{E}}\mathbf{C}^{-1}\} 2I_3(h'(I_3) + h''(I_3)I_3)\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\mathbf{C}^{-1}$$
$$- 2I_3h'(I_3)\mathbf{C}^{-1}\dot{\mathbf{E}}\mathbf{C}^{-1} \tag{84}$$

(B) In the case that **C** has only two distinct eigenvalues
The representation of $\mathbf{C}^{\alpha_p-1}$ may reduce to

$$\mathbf{C}^{\alpha_p-1} = \bar{\varphi}_{0p}\mathbf{I} + \bar{\varphi}_{1p}\mathbf{C} \tag{85}$$

By (72), the time rate of the stress has the representation

$$\dot{\mathbf{T}} = \sum_{p=1}^{N} \mu_p \{[(\alpha_p - 1)\mathbf{C}^{\alpha_p-2} - \bar{\varphi}_{1p}\mathbf{I}][\dot{\mathbf{E}} - (I_1^2 - 4I_2)(\mathbf{C}^2\dot{\mathbf{E}} - 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + 2\mathbf{C}\dot{\mathbf{E}}\mathbf{C} + \dot{\mathbf{E}}\mathbf{C}^2)] + 2\bar{\varphi}_{1p}\dot{\mathbf{E}}$$
$$- \frac{2\alpha_p}{3} I_3^{\alpha_p/3}\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\mathbf{C}^{-1} + 2I_3^{\alpha_p/3}\mathbf{C}^{-1}\dot{\mathbf{E}}\mathbf{C}^{-1}\}\ 2I_3(h'(I_3) + h''(I_3)I_3)\,\mathrm{tr}(\mathbf{C}^{-1}\dot{\mathbf{E}})\mathbf{C}^{-1}$$
$$- 2I_3h'(I_3)\mathbf{C}^{-1}\dot{\mathbf{E}}\mathbf{C}^{-1} \tag{86}$$

(C) In the case that three eigenvalues of **C** coalesce
Since $\hat{f}(\Lambda) = 3h'(\Lambda^3)\Lambda^2$, the stress rate yields

$$\dot{\mathbf{T}} = 2(6h'(\Lambda^3)\Lambda + 9h''(\Lambda^3)\Lambda^4)\dot{\mathbf{E}} \tag{87}$$

If we use the formula (48), the coefficients are quite complicated, even for the incompressible case (Basar and Itskov, 1998).

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## References

Ball, J.M., 1984. Differentiability properties of symmetric and isotropic functions. Duke Math. J. 51, 699–728.
Basar, Y., Itskov, M., 1998. Finite element formulation of the Ogden material model with application to rubber-like shells. Int. J. Numer. Methods Eng. 42, 1279–1305.
Chadwick, P., Ogden, R.W., 1971. A theorem of tensor calculus and its application to isotropic elasticity. Arch. Ration. Mech. Anal. 44, 54–68.
Dui, G., 1999. Some new representations for spin tensors. Mech. Res. Commun. 26, 529–534.
Dui, G., Ren, Q., Shen, Z., 1999. Time rates of Hill's strain tensors. J. Elast. 54, 129–140.
Dui, G., Ren, Q., Shen, Z., 2000. Conjugate stresses to Seth's strain class. Mech. Res. Commun. 27, 539–542.
Guo, Z.-H., Man, C.-S., 1992. Conjugate stress and tensor equation $\sum_{i=1}^{m} U^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$. Int J. Solids Struct. 29, 2063–2076.
Guo, Z.-H., Lehmann, Th., Liang, H., Man, C.-S., 1992. Twirl tensors and the tensor $\mathbf{AX} - \mathbf{XA} = \mathbf{C}$. J. Elast. 27, 227–245.
Gurtin, M.E., Spear, K., 1983. On the relationship between the logarithmic strain rate and the stretching tensor. Int. J. Solids Struct. 19, 437–444.
Hill, R., 1968. On the constitutive inequalities for simple materials-I. J. Mech. Phys. Solids 16, 229–242.

Hill, R., 1978. Basic aspects of invariance in solid mechanics. Adv. Appl. Mech. 18, 1–75.
Hoger, A., 1986. The material time derivative of logarithmic strain. Int. J. Solids Struct. 22, 1019–1032.
Hoger, A., 1987. The stress conjugate to logarithmic strain. Int. J. Solids Struct. 23, 1645–1656.
Itskov, M., 2000. On the theory of fourth-order tensors and their applications in computational mechanics. Comput. Meth. Appl. Mech. Eng. 189, 419–438.
Macmillan, E.H., 1992. On the spin of tensors. J. Elast. 27, 69–84.
Man, C.-S., 1995. Smoothness of the scalar coefficients in the representation $\mathbf{H}(\mathbf{A}) = \alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2$ for isotropic tensor functions of Class $C^\gamma$. J. Elast. 40, 165–182.
Man, C.-S., Guo, Z.-H., 1993. A basis-free formula for time rate of Hill's strain tensors. Int. J. Solids Struct. 30, 2819–2842.
Marsden, J.E., Hughes, T.J.R., 1993. Mathematical foundations of elasticity. Prentice-Hall, Englewood Cliffs, NJ.
Mehrabadi, M.M., Nemat-Nasser, S., 1987. Some Basic kinematics solutions for finite deformations of continua. Mech. Mater. 6, 127–138.
Ogden, R.W., 1974. On stress rates in solid mechanics with application to elasticity theory. Proc. Cambridge Philos. Soc. 75, 303–319.
Rivlin, R.S., Ericksen, J.L., 1955. Stress-deformation relations for isotropic materials. J. Ration. Mech. Anal. 4, 323–425.
Scheidler, M., 1991. Time rates of generalized strain tensors. Part I: component formulas. Mech. Mater. 11, 199–210.
Silhavý, M., 1997. The mechanics and thermodynamics of continuous media. Springer, New York.
Truesdell, C., Noll, W., 1965. The non-linear field theories of mechanics. In: Flügge, S. (Ed.), Handbuch der Physik, III/3. Springer-Verlag, Berlin.
Wang, W.B., Duan, Z.P., 1991. On the invariant representation of spin tensors with applications. Int. J. Solids Struct. 27, 329–341.
Xiao, H., 1995. Unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain. Int. J. Solids Struct. 32, 3327–3340.
Xiao, H., Bruhns, O.T., Meyers, A.T.M., 1998. Strain rates and material spins. J. Elast. 52, 1–41.